Notes on Spivak’s “Calculus on Manifolds”[1]

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Theory

1-1 (1)

$|x| = \sqrt{\sum_i^n{x_i^2}}$.
$\sum_i^n{x_i^2} = 0$ only if $x_i = 0$ for all x_i.
Therefore $|x| = 0$ only if $\emph{x} = \emph{0}$.

1-2 (3)

$x \cdot x = \sum_i^n{x_i^2} = |x|^2$. By (1-1) above,
$|x|^2 = 0$ only if $x = 0$, therefore $x \cdot x = 0$
only if $x = 0$.

(4)

By (3) above, $|x|^2 = x \cdot x$. We take the sqrt to
get $|x| = \sqrt{x \cdot x}$.

Problems

1-1

$x \cdot x \leq (\sum{|x_i|})^2$
$\sum{x_i^2} \leq (\sum{|x_i|})^2$ (with from i to n implied)
$\sum{x_i^2} \leq \sum{x_i^2} + C$, where C represents the extra
factors as a result of multinomial expansion.
We obtain $C \geq 0$, which is true since all
values $|x_i|$ in C are positive. ∎

1-2

For equality to hold, $\sum{x_i \cdot y_i} = |x| \cdot |y|$.
This is *NOT* the same as Theorem 1-1(2), as the LHS may be
negative. So equality holds when two conditions
are satisfied:

(1) x and y are linearly dependent
(2) <x,y> is nonnegative. ∎

1-3

Taking ?= to stand for an unknown comparison:
$|x-y|^2 ?= |x|^2 + |y|^2 + 2|x||y|$
$\sum{(x_i-y_i)^2} ?= \sum{x_i^2} + \sum{y_i^2} + 2|x||y|}$,
Expanding: $\sum{(x_i-y_i)^2} ?= \sum{x_i^2} + \sum{y_i^2}
- 2 \cdot \sum{x_iy_i}$
Canceling and simplifying: $-<x,y> ?= |x||y|}$
By Theorem 1-1(2) it is clear that if $-<x,y> \geq 0$,
$-<x,y> \leq |x||y|$. In the opposite case, $-<x,y> < 0$
and $|x||y| \geq 0$, so the comparison still holds. ∎

1-4

$||x|-|y|| = |\sqrt{\sum{x_i^2}} - \sqrt{\sum{y_i^2}}|$

[1] I cannot guarantee the accuracy of every solution. If you spot a mistake, let me know @ ian.arawjo - at - gmail - dot - com.

$|x-y| = \sqrt{\sum{x_i^2} + \sum{y_i^2} - 2 \cdot \sum{x_iy_i}}$
$|x-y|^2 = |x|^2 + |y|^2 - 2<x,y>$
$||x|-|y||^2 = |x|^2 + |y|^2 - 2|x||y|$
We're comparing: $||x|-|y||^2 ?= |x-y|^2$,
therefore $|x||y| =? <x,y>$ (NOTE the sign reversal!)
By Theorem 1-1(2), we know that $|x||y| \geq |<x,y>|$,
making $?=$ equal to $\leq$. Similarly, if $<x,y>$ is less than
0, then the comparison still holds.
Therefore $||x|-|y|| \leq |x-y|$. ∎

1-5

First, shift the axes so that x = 0 (origin). Then:
$|z| ?= |z-y| + |y|$ i.e.
$|z| - |y| ?= |z-y|$ i.e. (choose |z| to be \geq |y|)
$|z|^2 + |y|^2 - 2|z||y| ?= |z^2| + |y^2| - 2<z,y>$
$|z||y| =? <z,y>$ (same as 1-4 above)
Therefore $?=$ equals $\leq$ by Theorem 1-1(2). ∎
Geometric interpretation:
Here it is assumed that no two points are equal.
The points z,y,x form a triangle in R^n, $n \geq 2$.
By connecting points, we obtain side lengths |z-x|, |z-y|,
and |y-x|. The proved inequality states that no
side can be greater than the other two sides combined.

1-6

$\int_a^b{f} = \sum_{x=a}^b{f(x) \Delta x}$ as step
$\Delta x \to 0$. Hence:
$\int_a^b{fg} = \sum_{x=a}^b{f(x)g(x) \Delta x}$
$(\int_a^b{f^2})^{1/2} = \sqrt{\sum_{x=a}^b{f(x)f(x) \Delta x}}$
$(\int_a^b{g^2})^{1/2} = \sqrt{\sum_{x=a}^b{g(x)g(x) \Delta x}}$
If F and G stand for the infinite sums, then loosely speaking,
$|<F,G>| \leq |F||G|$ by Theorem 1-1(2).
(b)(c) omitted

1-7

(a) $|T(x)| = \sqrt{<T(x),T(x)>}$. We take T to be inner product
preserving, so that $<T(x),T(x)> = <x,x>$; then
$\sqrt{<x,x>} = |x|$. (inner product -> norm preserving)
Proving the converse:
$<Tx,Ty> = (|Tx + Ty|^2 - |Tx - Ty|^2) / 4$
$= (|T(x + y)|^2 - |T(x - y)|^2) / 4$
We take T to be norm preserving, so:
$<Tx,Ty> = (|x + y|^2 - |x - y|^2) / 4 = <x,y>$. ∎
(b) Say T was not 1-1. Then it could be that
|T(x)| = |x| and |T(a)| = |a|, for some $a \neq x$,
s.t. T(x) = T(a). It is certainly possible that a and x
are different vectors that give the same norm; consider
x = (1,0) and a = (0,1) where T(x) = T(a) = (1,0). However,
it is clear in that case that <Tx, Ta> \neq <x, a>. Generalizing,
we have <Tx, Tx> = <x, a> (since Ta = Tx), where a \neq x,
which violates inner product preservation. ∎
(Since T is a bijection, it must have an inverse.)

1-10

(omitted)

1-11

First: <(x,z),(y,w)> = \sum{x_i * y_i} + \sum{z_i * w_i} = <x,y> + <z,w>. (We could split the dot product since x and y, z and w are of the same length.)
Second: |(x,z)| = \sqrt{\sum{x_i^2} + \sum{z_i^2}} = \sqrt{|x|^2 + |z|^2}. ■

1-12

(omitted)

1-13

If <x,y> = 0, we have that |x+y|^2 - |x-y|^2 = 0 by polarization. Then |x+y|^2 = \sum{(x_i-y_i)^2} = |x|^2 + |y|^2 (since 2<x,y> = 0). ■

Subsets of Euclidean Space

Proof

By definition, a subset C of R^n is closed if R^n - C is open. Take a closed rectangle C in R^n, where C = [a1,b1] x … x [an, bn] and a_i < b_i by convention.
Now R^n - C = ((-inf,a1) U (b1, inf)) x … x ((-inf,an) U (bn, inf)).
Since R^n is composed of open intervals (open), C is closed. ■

Note on the definition of compactness:

"A set G is compact if every open cover C of the set contains a finite subcover U."

When I first read that statement, I was very confused. I thought: "Say G = (0, 1). Taking C = {G}, U = C, we have a finite subcover (one element) and so G is compact. (???)"

The key word in the definition is **every**. It's perhaps clearer if we invert it:

"A set G is noncompact if there exists an open cover C which contains only infinite subcovers."

Example: G = (0,1), C = {(1/n, 1-1/n) | n>1}.
C is infinite. Since C approaches (0,1) in the limit of n->+inf, effectively 'widening', no finite set will cover G, and thus we've proven that G is noncompact.

Alternative example: G = [0,1], C = {(-1+a, a) | a in (0, 2)}.
C is an infinite cover, yet (-0.01, 0.99) U (0.01, 1.01) covers G. This is evidence toward G being compact, but we'd have to

prove that there cannot exist an open cover with only infinite subcovers.

In plain english: to prove G is noncompact, find an open cover which approaches the boundary of G in its limit.

Problems

1-14

For every element x of an open set O in Euclidean space, there is an open rectangle A that contains x. Say we have open sets O1 and O2, with corresponding open rectangles A1 and A2. A1 U A2 is therefore a subset of the union O1 U O2, meaning that for each element x of O1 U O2, x is a member of A1, A2, or both. By definition, this makes O1 U O2 an open set. ∎

Similarly, take the union of any number of open sets $U = \cup_{i=1}^{n}{O_i}$, where n can stand for $\inf$. Then each O_i has an open rectangle A_i, and for each element x of U, $x \in A_i$ for some i. ∎

For intersection of two open sets $U = Oa \cap Ob$, we have $A = Aa \cap Ab$ where for all $x \in U$, $x \in A$ and where each Az contains all open rectangles of Oz. We must prove that the intersection of an open rectangle is open:

Choose any two rectangles from both Aa and Ab:
A1 = (a11, b11) x … x (a1n, b1n) and A2 = (a21, b21) x … x (a2m, b2m) for some $n,m >= 1$. Now for each dimension (a, b) in A_i there are three possibilities:

(1) m > n or n > m, so we drop the dimension;
(1) Either (a_1i, b_1i) and (a_2i, b_2i) are mutually exclusive, or
(2) intersect, in which case there is an open rectangle (a_2i, b_1i) that covers the intersection (we can choose the order of indices such that this is true).

Thus the intersection of two A_i's forms a new open rectangle where, freely choosing $n \leq m$,

$A_j = (a_21, b_11) x \ldots x (a_2n, b_1n)$

and if any dimension is the null set {} (an open set), A_j = {}.
From this, $Aa \cap Ab$ can be reconstructed as the union of the intersection of all open rectangles between them. Thus for every element x in U, x is a member of an open rectangle in A. ∎

On the contrary, infinitely many open sets can converge to a number. Consider a set which "shrinks" in its limit, converging to a value smaller than its starting value. For instance, O = {(1/n, 2+1/n) | n \in +Z} converges to the interval [0, 2] as n goes to infinity. Intersection of all O's elements yields [0, 2], a closed set. ∎

1-15

Taking r = 0 yields the null set {}. If r > 0, then by the properties of the real numbers, there exists an 0 \leq x < r. Take a = 0 to maximize x. Now x equals plus or minus r-e for some e approaching zero. Thus for the given r we have (-r, r), an open set. Since the union of infinite open sets is open (1-14), {x \ in R^n : |x-a| < r} is open. ∎

1-16

(i) interior: |x| < 1, exterior: |x| > 1, boundary: |x| = 1.
(ii) interior: {}, exterior: |x| != 1, boundary: |x| = 1.
(iii) interior: {}, exterior: at least one x_i
is irrational, boundary: each x_i is rational

1-17

A = {point(x, x) | 0 < x < 1} (i.e. an open diagonal line)
* I think this is what the question is asking for?

1-18

A = {(ai,bi) | 0 < a < b < 1 and rational r \in (ai, bi)}.
It is obvious that this union results in the open set (0, 1).
Therefore the boundary of A must be the limits 0 and 1,
i.e. [0,1] - A.

1-19

0 and 1 are rational. We must show that there are no 'gaps' containing only irrational numbers in the interval (0, 1). Suppose such an interval [a,b] existed, where 0 < a < b < 1 and a,b are irrational. By the density of the reals, there exists a rational number a < r < b. So the interval [a,b] cannot exist, and [0,1] must be contained in A. ∎

1-20

$U \subset R$ and U is compact, meaning that every open cover O of U contains a finite subcover A. Say U was open ($v \neq w$). For each $U_i = (v,w)$ in U, we can form an infinite cover:

$O = {(v+(v+w)/n, w-(v+w)/n) | n \geq 2 }$

which approaches (v,w) in the limit n->inf. Therefore U cannot be open. Moreover, U is bounded by $\cup_i{{v_i, w_i}}.$

By Corollary 1-6, $X \subset R^n$ is compact if each $X_i \subset R$ is. By the above reasoning, if X_i is compact, X_i is closed and bounded. Therefore X is closed and bounded. ∎

1-21 (a) Suppose there isn't such a number. Then |y - x| = 0. For that to be true, x = y. But if x = $y \in A$ then $x \in A$, a contradiction.
(b) (c) omitted

1-22 If $C \in R^n$ is a compact subset of the open set U, then C is closed and bounded (Problem 1-20). Thus U-C is open and cannot be empty. Create the set Y as follows: For each c on the boundary of C, construct an open rectangle within U (this is possible as U is open). Then D = C + Y + boundary(Y), and $C \subset D^o, D \subset U$. ■

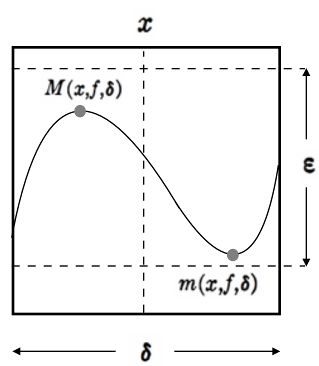
Functions and Continuity

Note on Theorem 1-11 (Proof)

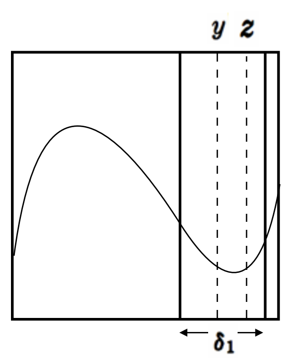
Spivak's notation can become a bit cumbersome, so here's a visual description of the two cases.

The first case is trivial: imagine removing a bounded rectangle from R^n. The resulting space would be open.

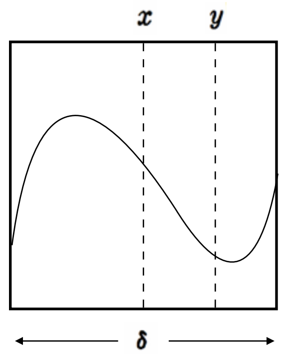
The second case appears tricky. Keep in mind the fact that M(…)-m(…) stands for (roughly) the maximum value - minimum value of the function inside a given interval. If we visualize this interval as a square,



consider a value y such that |x-y| < delta:



In other words, y is 'inside the square.' Now since the total interval is open (we've established this with M(…)-m(…) < e), we can find an even smaller interval:



Since M(…)-m(…) inside this smaller interval is less than e, o(y,f) < e, proving C \subset R^n - B.

Problems

Diversion into Spivak's Calculus

Why the diversion? Well, it's been awhile since I learned Calc I. Like most people, I memorized the derivative / integral identities and virtually forgot about how to derive the fundamental theorem of calculus. Coming from the side of open balls in R^n, I see now how intertwined the concept of a limit is with the boundary of an open interval. But to know I am answering these problems satisfactorily, we must go off on a tangent into Spivak's "Calculus," chapters 5 and 6. (I promise that this tangent will reveal its further use shortly!)

Chapter 5: Limits

Definition: The function f approaches the limit L near a if for every e > 0 there is some d > 0 such that, for all x, if 0 < |x-a| < d, then |f(x)-L| < e.

("This definition is so important, everything we do from now on depends on it." (p. 79))

Properties:
lim fg = lim f * lim g
lim (f+g) = lim f + lim g
lim 1/g = 1/(lim g) as long as g /neq 0

Chapter 6: Continuity

Definition: The function f is 'continuous at a' if the limit of f(x) as x approaches a equals f(a).

In other words if f is everywhere continuous, then any limit of f at a can be replaced by f(a).

Properties:
If f,g are continuous at a, then:
f + g is continuous at a
fg is continuous at a

if g(a) \neq 0, 1/g is continuous at a

Composition: If g is continuous at a and f is continuous at g(a), then f(g(x)) is continuous at a.

Now we are ready to tackle continuity problems with some clarity behind our assumptions.

1-23

A) If lim f(x) = b (as x approaches a), show that
lim f^i(x) = b^i for i from 1..m.

By definition on p. 11, f(x) = (f^1(x), … , f^m(x)).

lim f(x) = lim (f^1(x), … , f^m(x)) = (b_1, …, b_m)
= lim ((f_1(x), … , 0) + … + (0, … , f^m(x)))
= lim (f_1(x), … , 0) + … + lim (0, … , f^m(x)) (prop)
which means for each i,
lim (… f_i(x) …) = (… b_i …)
since lim 0 = 0. ∎

B) omitted (the derivation is similar)

1-24

A) lim f(x) = lim (f_1(x), … , 0) + … + lim (0, … , f^m(x))
which means for each i,
lim (… f_i(x) …) = (… f(a) …)
for f to be continuous at a. ∎

B) omitted

1-25

If a function h is continuous, then T(h) is a linear combination of h_i's. By the additivity and homogeneity properties of limits, T(h) must also be continuous (see 1-24).

1-26

(a) A straight line can be defined by the set of points (x, ax).
Let C = R^2 - A =
{(x,y) in R^n : x \leq 0 and (y \leq 0 or y \geq x^2)}
Now for each line {(x, ax)}, if a or x is \leq 0, the point is in C. For (x, ax) > (0,0), we must show that ax > x^2 over some interval (0, d). This is the same as saying x < a. Since a > 0, we can find an x such that 0 < x < a. Taking d=a completes the proof. ∎

(b) omitted

1-27

**(to do: prove f is continuous)
If f(x) is continuous then the limit as x->a is f(a) = 0. Pick some e \in R^n, r>|e|>0, and produce x = (a + e). Now f(a+e) = |e|. Form a new vector d \in R^n such that r > |d| > |e| (property of the real numbers). Now f(a+e+d) = |d+e| < r. Since we can do this ad infinitum, there must exist an open interval bounded by magnitude r.

Chapter 2: Differentiation

Note on Theorem 2-3

Spivak uses the notation Df(a) = f here, which considerably confused me. The terseness of the statement is not strictly justified; it would, in my opinion, be much clearer if Spivak used Δx or h.

The source of the confusion stems from Df(a)'s definition on page 17, where Spivak uses the same variables x,y to mean two different things:

"For the moment let us consider the function f:R^2->R defined by f(x,y) = sin x. Then Df(a,b) = $\boldsymbol{\lambda}$ satisfies $\boldsymbol{\lambda}$(x,y) = (cos a)x."

Since he's defining the function, technically Spivak can choose whichever free variables he wishes; however it would help considerably if the chosen variables were consistent with their contextual meanings, e.g. $\boldsymbol{\lambda}$(h,k).

Note that Df(a) = f doesn't mean the derivative itself is f(a) (at least in the classic sense), but f(Δx).